Pergamon

J. Appl. Maths Mechs, Vol. 58, No. 2, pp. 223–234, 1994

0021–8928/94 $24.00+0.00

0021–8928(94)E0045-C

COMPARISON OF THE STABILITY PROPERTIES OF TWO AUTOROTATING MOTIONS†

V. A. PRIVALOV and V. A. SAMSONOV

Moscow

(*Received* 16 *February* 1993)

The properties of the steady free fall of an autorotating rigid body in a resistant medium are compared with those of the corresponding permanent rotation of a mock-up of the body in a wind-tunnel. The relative motions of the body and the medium are the same in both these steady motions, but the body has a different number of degrees of freedom.

The domain of asymptotic stability of steady fall is constructed in parameter space and compared with the domain of stability of permanent rotation [1]. It has been established that, regardless of the difference in the dynamics of the bodies, the boundaries of the domains of stability coincide in part. Characteristic parameters are determined to enable a measure of the difference between the domains to be defined. Stability is analysed using a geometrical interpretation of the conditions, established in [2], that the roots of a certain third-degree polynomial with complex coefficients have negative real parts.

As no comparison theorems (like the Kelvin–Chetayev theorems on the effect of dissipative forces) are available for dynamical systems that differ in their dimensions, an examination of specific example may be of some methodological interest. This is particularly true in view of the fact that, although some experience has been gained in experimenting with moving mock-ups in wind-tunnels [3], the question of the correspondence between these mock-ups and their prototypes in free flight still awaits a thorough investigation.

1. MATHEMATICAL MODEL OF THE MOTION

Consider a dynamically symmetrical body falling freely in an undisturbed atmosphere. In addition to its weight $M\mathbf{g}$, the body is subjected to a distributed system of forces exerted by the incident flow. As in [1], we shall regard this force as concentrated on four vanes, whose location on the body causes autorotation round the body's axis of dynamical symmetry. Our problem is whether a motion of uniform vertical descent exists during which the body rotates at a constant angular velocity around a vertically oriented axis of symmetry, and, if it does, whether such motion is stable.

The action of the medium on the body will be simulated by a quasistatic model [4]. Under certain conditions this yields a satisfactory description even of unsteady motion [5, 6].

Let us assume that the system of forces acting on vane j is equivalent to the resultant $\mathbf{R}_j$ of two mutually perpendicular components: the drag force $\mathbf{W}_j$ and the analogue $\mathbf{Y}_j$ of the lift. The point O_j at which the force $\mathbf{R}_j$ is applied (the centre of pressure) is assumed to be a fixed point on the vane j. The velocity of O_j is then given by

†*Prikl. Mat. Mekh.* Vol. 58, No. 2, pp. 37–48, 1994.

$$\mathbf{V}_j = \mathbf{V} + \boldsymbol{\Omega} \times \mathbf{r}_j \tag{1.1}$$

where Ω is the angular velocity of the body and r_j the radius-vector of O_j relative to the centre of mass, which is moving at a velocity **V**.

We write the vectors $\mathbf{W}_j$ and $\mathbf{Y}_j$ in the form

$$\mathbf{W}_j = -\tfrac{1}{2}\rho c_x(\alpha_j) S V_j \mathbf{V}_j \tag{1.2}$$

$$\mathbf{Y}_j = \tfrac{1}{2}\rho c_y(\alpha_j) S(\mathbf{V}_j \times \mathbf{n}_j) \times \mathbf{V}_j / \cos\alpha_j \tag{1.3}$$

where ρ is the density of the atmosphere, S the characteristic area of the vane, and $c_x(\alpha_j)$, $c_y(\alpha_j)$ are dimensionless aerodynamic coefficients, which are functions of the angle of attack α_j of the vane j, i.e. the angle between the vector $\mathbf{V}_j$ and the plane of the vane. The vector $\mathbf{n}_j$, the normal to the latter plane, determines the orientation of the vane in a frame of reference attached to the body.

It can be seen that the resistance force $\mathbf{W}_j$ points in the direction opposite to that of the velocity $\mathbf{V}_j$ of the centre of pressure O_j relative to the stationary medium, while $\mathbf{Y}_j$ is perpendicular and lies in the plane of the angle of attack α_j, formed by the vectors $\mathbf{V}_j$ and $\mathbf{n}_j$.

We shall assume that the centres of pressure O_j and the body's centre of mass lie in a single plane orthogonal to the axis of the symmetry of the body, at a distance r from the axis and forming the vertices of a square.

The sum of the forces $\mathbf{W}_j + \mathbf{Y}_j$ applied to vane j produces a moment $\mathbf{M}_j$ about the centre of mass

$$\mathbf{M}_j = \tfrac{1}{2}\rho c_x(\alpha_j) S V_j \{k(\alpha_j)(\mathbf{r}_j \times \mathbf{n}_j) V_j / \cos\alpha_j - (\mathbf{r}_j \times \mathbf{V}_j)[1 - k(\alpha_j)\operatorname{tg}\alpha_j]\} \tag{1.4}$$

$$k(\alpha_j) = c_y(\alpha_j) / c_x(\alpha_j)$$

where $k(\alpha_j)$ is the lift–drag ratio of the vane.

The moment about O_j of the forces exerted by the medium will be ignored. Thus, the forces and their moments depend on the angles that determine the orientation of the vector **V** relative to the body, on the body's angular velocity and on the differences between the velocities and angles of attack of the vanes. Further allowance for the difference between the velocities at different points of the same vane does not essentially alter the structure of this model of the action of the medium; its contribution will be less the smaller the vanes compared with the dimensions of the body.

To describe the motion of the body we shall use a system of coordinates $Oxyz$ whose z axis coincides with the body's axis of dynamic symmetry, while the x and y axes are placed in sueh a way that the centres of pressure O_j lie on them. We then have the following formulae for the momentum **L**, the angular momentum **G** relative to the centre of mass, and the vectors $\mathbf{r}_j$ and $\mathbf{n}_j$

$$\mathbf{L} = \mathbf{l}MV_x + \mathbf{m}MV_y + \mathbf{n}MV_z \tag{1.5}$$

$$\mathbf{G} = \mathbf{l}A\Omega_x + \mathbf{m}A\Omega_y + \mathbf{n}C\Omega_z \tag{1.6}$$

$$\mathbf{r}_j = \mathbf{l}r\cos\psi_j + \mathbf{m}r\sin\psi_j \tag{1.7}$$

$$\mathbf{n}_j = -\mathbf{l}\cos\beta\sin\psi_j + \mathbf{m}\cos\beta\cos\psi_j + \mathbf{n}\sin\beta \tag{1.8}$$

where M, A and C are the mass and equatorial and polar moments of inertia of the body, **l**, **m** and **n** are unit vectors on the x, y, z axes, respectively ψ_j is an angle whose values are $\psi_1 = 0$, $\psi_2 = \pi/2$, $\psi_3 = \pi$, $\psi_4 = -\pi/2$ and which determines the position of the point O_j, β is the angle between the normal $\mathbf{n}_j$ and the plane xOy, i.e. the angle between the vane j and the appropriate coordinate axis, V_x, V_y, V_z are the projections of the vector $\mathbf{V}$, and Ω_x, Ω_y, Ω_z the corresponding projections of the angular velocity $\mathbf{\Omega}$ of the body.

The orientation of the body (hence also of the system $Oxyz$) in space is determined by the angles φ, θ, γ, where φ and θ are the Krylov angles, which characterize the position of the z axis (the axis of symmetry of the body) in a system of coordinates rotating at angular velocity $\dot{\gamma}$ about a vertical axis through the body's centre of mass.

The passage from a non-rotating system of coordinates to the system $Oxyz$ moving with the body involves a sequence of rotations that does not introduce terms modulated by the angle of spin in the formulae for the external forces and moments. This in turn implies that the equations of the perturbed motion will not involve any periodic coefficients so that these equations will constitute a stationary system.

The projections of the angular velocity $\mathbf{\Omega}$ are

$$\Omega_x = \dot{\Theta} - \dot{\gamma}\sin\varphi, \quad \Omega_y = \dot{\gamma}\cos\varphi\sin\Theta + \dot{\varphi}\cos\Theta$$
$$\Omega_z = \dot{\gamma}\cos\varphi\cos\Theta - \dot{\varphi}\sin\Theta.$$

By (1.1), the projections of the velocities $\mathbf{V}_j$ of the pressure centres O_j are given by

$$V_{xj} = V_x - r\Omega_z \sin\psi_j, \quad V_{yj} = V_y + r\Omega_z\cos\psi_j$$
$$V_{zj} = V_z + r(\Omega_x \sin\psi_j + \Omega_y\cos\psi_j)$$

It is clear from our account that the equations of motion are projected onto axes rigidly attached to the body. This is because the expressions for the forces and moments are most easily obtained as functions of the projections of the velocities of the centres of pressure on such axes. In what follows, after deriving linearized equations, we shall transform to a comoving frame of reference.

2. STEADY MOTION

Let us assume that the body is moving in such a way that its centre of mass descends vertically while the body itself rotates about the vertically oriented axis of symmetry. The projections of the angular velocity $\mathbf{\Omega}$ and the linear velocity $\mathbf{V}$ of the body and the angles that determine the orientation of its axis of symmetry are

$$\Omega_x = \Omega_y = 0, \quad V_x = V_y = 0, \quad \varphi = \theta = 0 \tag{2.1}$$

In this case the moduli of the velocities $\mathbf{V}_j$ of the centres of pressure O_j relative to the medium are equal, as are the angles of attack α_j; their values are respectively V_r and α. These quantities satisfy the kinematic relations

$$V_r = [V_z^2 + (r\Omega_z)^2]^{1/2}, \quad r\Omega_z = -V_r\sin(\alpha - \beta),$$
$$V_z = -V_r\cos(\alpha - \beta) \tag{2.2}$$

Substituting expressions (1.1), (1.7) and (1.8) for $\mathbf{V}_j$, $\mathbf{r}_j$, $\mathbf{n}_j$ into formulae (1.2)–(1.4) for the aerodynamic forces $\mathbf{W}_j$, $\mathbf{Y}_j$ and their moments $\mathbf{M}_j$ and transforming in accordance with (2.1) and (2.2), we obtain the projections of the forces and moments of the overall action of the medium

$$R_x = 0, \quad R_y = 0, \quad R_z = -2\rho c_x(\alpha)SV_rV_z[1 - k(\alpha)\operatorname{tg}(\alpha - \beta)]$$

$$M_x = 0, \quad M_y = 0, \quad M_z = -2r\rho c_x(\alpha)SV_rV_z[k(\alpha) + \operatorname{tg}(\alpha - \beta)]$$

If $\alpha=\alpha_0$ satisfies the transcendental equation

$$k(\alpha_0)+\mathrm{tg}(\alpha_0-\beta)=0 \tag{2.3}$$

then the moment M_z vanishes. But if, in addition, the vertical projection is $V_z=V_{z0}$, where V_{z0} is a root of the equation

$$Mg-2\rho c_x(\alpha_0)SV_{z0}^2/\cos^3(\alpha_0-\beta)=0 \tag{2.4}$$

then the sum of all the external forces applied to the body will also vanish. In that situation the angle of attack α_0 and the velocities V_{z0} will correspond to constant values of the angular velocity Ω_0 of the body and the velocity V_{r0} of the centre of mass

$$r\Omega_{z0}=V_{z0}\,\mathrm{tg}(\alpha_0-\beta),\quad V_{r0}=[V_{z0}^2+(r\Omega_{z0})^2]^{1/2}$$

Whenever the sum of all applied forces and moments vanishes, one particular solution of the equations of motion will be vertical uniform fall at a velocity V_{z0}, with the body rotating at an angular velocity Ω_{z0} about a vertical axis of symmetry.

The angle of attack α_0 in such situations of steady fall satisfies the same transcendental equation as in the case of permanent rotation in flow in a wind-tunnel [1], about an axis of symmetry parallel to the flow direction. Consequently, a rotating body falling steadily in a medium, as described by Eqs (2.1), (2.3) and (2.4), may be compared with the same body, but with a stationary centre of mass, autorotating in a flow. The relative motion of the body in the medium is the same in both cases.

3. STABILITY TO PERTURBATIONS OF THE VERTICAL COMPONENTS OF THE LINEAR AND ANGULAR VELOCITIES

To analyse the stability of the steady motion (2.1), (2.3) and (2.4), we will set up the equations of perturbed motion. To do this, we linearize the expressions for the applied forces (1.2), (1.3), the moments (1.4), and the total derivatives of the momentum (1.5) and angular momentum (1.6). This gives a set of equations consisting of two independent subsystems. The increments ΔV_x, ΔV_y, $\Delta\Omega_x$, $\Delta\Omega_y$, $\Delta\varphi$, $\Delta\theta$ satisfy one subsystem. The other consists of two equations in ΔV_z and $\Delta\Omega_z$

$$\begin{aligned}
&M\Delta\dot V_z=-2\rho c_xSV_{r0}[\zeta(r\Delta\Omega_z+k\Delta V_z)+2\Delta V_z(1+k^2)]\\
&C\Delta\dot\Omega_z=-2r\rho\eta c_xSV_{r0}(r\Delta\Omega_z+k\Delta V_z)\\
&\xi=(c_xc_x'+c_yc_y')/(c_x^2+c_y^2)\\
&\eta=1+k'\cos^2(\alpha-\beta),\quad \zeta=\xi-2k
\end{aligned}\tag{3.1}$$

Here and below c_x, c_y, k, c_x', c_y', k' are the values of the functions $c_x(\alpha)$, $c_y(\alpha)$, $k(\alpha)$ and their derivatives with respect to α at $\alpha=\alpha_0$.

Introducing a new independent variable $\tau=2c_x\rho SV_{r0}t/M$ in system (3.1), we obtain the characteristic equation of the transformed system

$$\varkappa^2+\varkappa(\delta+\eta/P)+2\eta(1+k^2)/P=0$$

$$\delta=2+k\xi,\quad P=C/(Mr^2)$$

The velocity increments ΔV_z and $\Delta\Omega_z$ will tend to zero if

$$\eta > 0, \quad \delta + \eta/P > 0 \tag{3.2}$$

Let us check these inequalities against the conditions for the steady value of the angular velocity of rotation of the body about a stationary axis parallel to the flow to be asymptotically stable [1]. It can be seen that the first inequality of (3.2) represents the stability of steady rotation of the body in the flow. For steady fall at velocities V_{z0}, Ω_{z0} to be asymptotically stable, one more condition is necessary: the second inequality of (3.2). This inequality, unlike the first, involves not only the aerodynamic parameters but also P, which characterizes the mass distribution in the body. As P decreases, the conditions (3.2) for the asymptotic stablity of steady fall approach the first asymptotic stability condition (3.2) for rotation of a body in a flow at constant angular velocity.

If the vanes are shaped like flat plates, the motion of the body will have the following property [7, 8]. The functions $c_x(\alpha)$, $c_y(\alpha)$ behave typically in such a way that, for angles of attack not exceeding $\pi/6$ the quantity ξ remains positive. Hence the first inequality of (3.2) is a necessary and sufficient condition for stability in this range of α_0. The second inequality imposes no further restrictions.

4. STABILITY OF THE VERTICAL ORIENTATION OF THE AXIS OF SYMMETRY AND THE VELOCITY VECTOR OF THE CENTRE OF MASS IN FREE FALL

To simplify the system of equations for small oscillations of the axis of symmetry and small variations in the direction of the velocity of the centre of mass, we will introduce complex variables λ, ω, u and transform to an attached system of coordinates by the formulae

$$\lambda = (\Delta\theta + i\Delta\varphi)\exp(-i\Omega_{z0}t), \quad \omega = (\Delta\Omega_x + i\Delta\Omega_y)\exp(-i\Omega_{z0}t)$$
$$u = (\Delta V_x + i\Delta V_y)\exp(-i\Omega_{z0}t)$$

As a result, the linearized equations may be brought to the form

$$\dot{\lambda} = \omega, \quad A\dot{\omega} - iC\Omega_{z0}\omega = r\rho c_x SV_{r0}(\zeta u - r\omega\delta)$$
$$M(\dot{u} - iV_{z0}\omega) = -i\lambda Mg + \rho c_x SV_{r0}[r\omega k\eta - u(1 + \eta - k\,\mathrm{tg}\,\alpha)]$$

The number of dimensions of the parameter space of this system may be minimized by normalizing the variables. The unit of the new independent variable will be $T = A(r^2\rho c_x SV_{r0})^{-1}$, the unit of angular velocity $1/T$, and the unit increment of linear velocity $|V_{z0}|$. Clearly, T is proportional to the characteristic time for the so-called fast variation of the angular velocity ω [9].

Transforming the equations of motion in accordance with (2.3) and (2.4), we see that the normalized variables u and ω satisfy the system

$$\dot{\lambda} = \omega, \quad \dot{\omega} + \omega(\delta + ib_1) - uN\zeta\cos(\alpha_0 - \beta)/c_x = 0$$
$$\dot{u} + \varepsilon u(1 + \eta - k\,\mathrm{tg}\,\alpha) + \omega\{i - \eta\varepsilon kc_x/[N\cos(\alpha_0 - \beta)]\} + 2i\varepsilon\lambda/\cos^2(\alpha_0 - \beta) = 0 \tag{4.1}$$
$$b_1 = -kCN\cos(\alpha_0 - \beta)/(Ac_x)$$

The coefficients of system (4.1) are functions of the aerodynamic parameters at $\alpha = \alpha_0$, and also of the three quantities C/A, $\varepsilon = A/(Mr^2)$, $N = A/(\rho r^3 S)$, which involve the mass and geometrical characteristics of the body.

The characteristic equation of system (4.1) is

$$\mu^3+(a_1+ib_1)\mu^2+(a_2+ib_2)\mu+ib_3=0 \tag{4.2}$$

where

$$a_1=\delta+\varepsilon(1+\eta-k\,\mathrm{tg}\,\alpha),$$
$$a_2=\varepsilon[\delta(1+\eta-k\,\mathrm{tg}\,\alpha)-k\eta\varepsilon\zeta]$$
$$b_2=[\zeta-\varepsilon k(1+\eta-k\,\mathrm{tg}\,\alpha)]N\cos(\alpha_0-\beta)/c_x,$$
$$b_3=2\varepsilon N\zeta/[c_x\cos(\alpha_0-\beta)]$$

Necessary and sufficient conditions for the steady fall under consideration to be asymptotically stable will be obtained as conditions for the real parts of the roots of Eq. (4.2) to be negative. These conditions impose restrictions on certain determinants of the coefficients of Eq. (4.2), which may be expressed in the form [2]

$$a_1>0 \quad b_2b_3>0 \tag{4.3}$$

$$b_3(a_1b_1b_2^2+a_1^2a_2b_2-a_1^3b_3-b_2^3)>0 \tag{4.4}$$

5. ANALYSIS OF THE RESTRICTIONS ON THE COEFFICIENTS OF THE CHARACTERISTIC EQUATION

The characteristic equation, which is of degree 3, may always be reduced to the form (4.2), in which the free term is purely imaginary. In our present situation this is a consequence of the transformation to an attached system of coordinates. The stability conditions (4.3) and (4.4) impose restrictions on the five real coefficients a_i and b_i. One can form three functions of these coefficients such that conditions (4.3) and (4.4) admit of an easily understood geometrical interpretation.

Such functions may be constructed in more than one way. The specific construction selected is influenced, on the one hand, by the nature of the dependence of the coefficients a_i and b_i on the physical parameters of the problem and, on the other, by the admissible range of variation of these parameters, and hence also of the coefficients a_i and b_i themselves. To make the selection in this case, we use our initial assumption concerning the geometry of the vanes.

We shall assume that the vanes are similar to flat plates and that their angle β is positive and bounded by a certain number $\beta_*<\pi/2$. The specific bound β_* is chosen so as to guarantee that the transcendental equation (2.3) should have a unique solution. The solution may fail to be unique because at β values near $\pi/2$ the range of angles of attack α_0 of the solutions of Eq. (2.3) moves into the domain $\pi/6<\alpha_0<\pi/2$, where the lift–drag ratio $k(\alpha)$ behaves like the function $-\mathrm{tg}(\alpha-\pi/2)=\mathrm{ctg}\alpha$ and Eq. (2.3) becomes an identity [5]. Such behaviour of $k(\alpha)$, judging from experimental results [7, 8], is typical for the aerodynamics of flat plates, probably because of how they behave in flows at large angles of attack.

With β_* suitably chosen, the solution of Eq. (2.3) usually lies in an interval in which $k(\alpha)$ increases monotonically and is unique. Moreover, as a rule ξ and $1-k\,\mathrm{tg}\alpha_0$ are positive in that interval. As a result, conditions (3.2) and (3.3) for asymptotic stability of the steady values Ω_{z0}, V_{z0} are satisfied, but in addition the coefficients a_1 and a_2 remain positive.

Thus, for this vane geometry, some of the stability conditions are satisfied and the coefficient α_2 is positive. This makes it possible to form three real-valued functions $X=b_2/a_1\sqrt{a_2}$, $Y=b_3/a_2\sqrt{a_2}$, $Z=-b_1/\sqrt{a_2}$ from the coefficients a_i and b_i, which moreover have no singularities. Accordingly, we have a three-dimensional system of coordinates XYZ.

Characteristic equations of type (4.2) arise when investigating mechanical systems with three degrees of freedom possessing a certain symmetry. Therefore, the stability domain (SD) in the space of parameters X, Y, Z, which are related to the coefficients of Eq. (4.2), presents a special, universal geometrical picture, which will now be described.

For simplicity, we will consider the half-space $Z > 0$ (in particular, in our specific situation, $b_1 < 0$). In the half-space $Z > 0$, the inequality $b_2 b_3 > 0$ leaves only two quadrants for the SD, in the first of which $X > 0$, $Y > 0$, and in the second, $X < 0$, $Y < 0$. Condition (4.4) is satisfied for the points of these quadrants, which lie respectively below and above the surface E whose equation is

$$Y = X - ZX^2 - X^3 \tag{5.1}$$

Consequently, the SD in XYZ space is the non-empty set between the XZ plane and the surface E where $Z \geqslant 0$.

Let us consider some properties of this surface E (Fig. 1). For any fixed Z, the quantity Y is a cubic function of X. The cubic parabolas in which E cuts the plane $Z = 0$ and all planes parallel to $Z = 0$ intersect the X axis at $X = 0$ and at two other points, defined by the roots of the quadratic equation

$$1 - ZX - X^2 = 0 \tag{5.2}$$

These roots are real and of different signs, i.e. the points at which the cubic parabola (5.1) intersects the X axis lie on different sides of the origin. At $Z = 0$ the roots equal 1 in absolute value. As Z increases the positive root tends to zero, and the negative root to $-Z$. Hence it follows that the surface E cuts the plane $Y = 0$ on the Z axis and on the hyperbola (5.2) in the XZ plane, whose asymptotes are the Z axis and the straight line $Z = -X$.

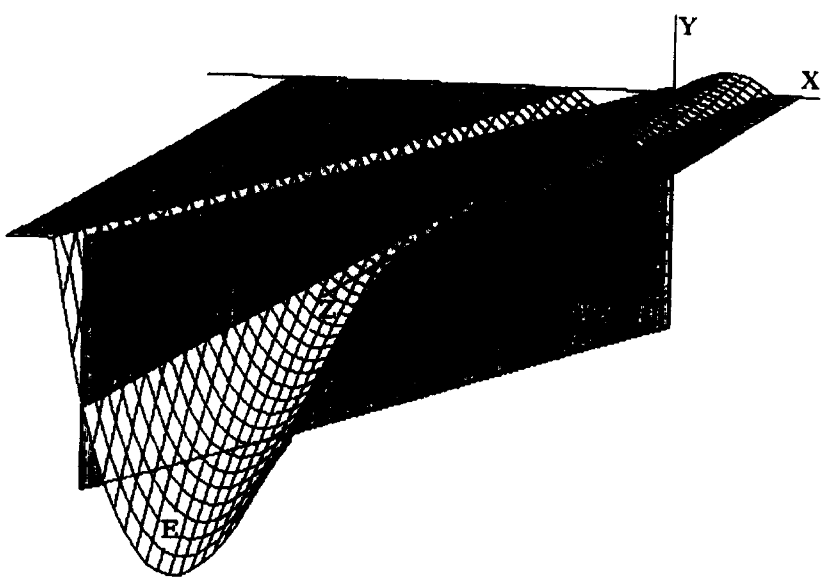

Fig. 1.

6. COMPARISON OF THE STABILITY DOMAINS OF STEADY FALL AND PERMANENT ROTATION

The stability domain of permanent rotation (SDPR) [1] was constructed in the k, ξ plane; part of its boundary runs along the straight line $\xi = 2k$. Part of the boundary of the stability domain of steady fall (SDSF) lies in the plane $Y = 0$. As these equations are clearly equivalent, the boundaries of the stability domains in both motions have sections that lie in the plane $Y = 0$. However, while the SDSF is doubly connected, i.e. part of it lies above the plane $Y = 0$, where $X > 0$, and part below, where $X < 0$, the SDPR is simply connected, lying below the straight line $\xi = 2k$ in the k, ξ plane, or what is the same, below the plane $Y = 0$, where $X < 0$. Herein lies the difference between the stability properties of the motions: the permanent rotation corresponding to the region $Y > 0$, i.e. $\xi = 2k$, is always unstable. The root of this instability is the destabilizing effect of non-conservative positional forces (such as the Magnus force). In free fall this factor may be neutralized by the link between the degrees of freedom of rotational and translational motion. Thus, in a sense, the SDSF is larger than the SDPR.

Besides the common section of the boundary in the plane $Y = 0$, the other parts of the boundaries of the SDs of the two motions are not generally the same. In permanent rotation the other part of the boundary forms the curve $\xi = 2k(1 - C/A)/(1 + k^2C/A)$ in the k, ξ plane [1]. This equation is equivalent to $a_1b_1 = b_2$, which is equivalent in XYZ space to the equation of the vertical plane $X + Z = 0$ (see Fig. 1). A necessary condition for the permanent rotation to be stable is, in particular, that $a_1b_1 - b_2 < 0$. Hence the SDPR lies beneath the plane $Y = 0$, where $X \leqslant 0$, and to the right of the plane $X + Z = 0$, i.e. where $X > -Z$.

It can be seen that the plane $X + Z = 0$ cuts the surface E along the straight line $X = Y$, $Z = -X$. Consequently, the plane $Y + Z = 0$ cuts the part of the SDSF lying beneath the plane $Y = 0$ into two parts. Hence it is clear that not only when $Y > 0$ but also when $Y < 0$ there is a part of the SDSF lying outside the SDPR.

However, the part of the SDPR situated beneath the surface E lies outside the SDSF.

7. PARAMETRIC ANALYSIS

The three numbers X, Y, Z determine the position of the point with these coordinates, say B, relative to the SD. Hence analysis of X, Y, Z as functions of the parameters N, ε, C/A may provide a qualitative picture of the effect of these parameters on the stability properties of the steady motions under consideration.

Let us consider the role of the parameter $N = A/(\rho r^3 S)$, which is known as Newton's number. It can be seen that the absolute values of X, Y, Z are directly proportional to N. Thus, as N varies, the point B moves in XYZ space along a ray through the origin, in a direction equal to that of the vector with projections X/N, Y/N, Z/N, which is independent of N. Hence it follows that, depending on the shape of the boundary of the SD and the direction of the aforementioned vector, variation of N may have a two-fold effect.

For example, if the point B for some value of N lies in the component of the SDSF with $Y > 0$, increasing N will always cause a loss of stability. The same is true if the point B lies in the component of the SD where $Y < 0$ and $Z < -X$. But if the point B lies at the intersection of the stability domains, an increase in N will have no destabilizing effect.

The parameter $\varepsilon = A(Mr^2)$ characterizes the concentration of the body's mass relative to its "aerodynamic size" r. At small values of ε any change will have the opposite effect to that of a change in N, since X, Y, Z are inversely proportional to $\sqrt{\varepsilon}$.

For infinitely small ε or infinitely large N the stability condition (4.4) for free fall is equivalent to the corresponding stability condition for permanent rotation. In particular, in XYZ space this means that as Z increases the intersection of the stability domains, which lies in a plane $Z = \text{const}$, will expand and the parts of the SDSF outside the intersection will contract. In addition, if ε tends to zero because of the increase in Mr^2, the number $C/(Mr^2)$ will also decrease. As a result, all the stability conditions for steady fall degenerate into the stability conditions for permanent rotation.

Thus, the parameters ε and N may serve as a measure of the difference between the stability conditions of the two motions.

We will now consider the effect of a change in the ratio C/A of the moments of inertia on the stability of steady fall, assuming that the other parameters remain fixed.

The quantities X and Z are linear functions of C/A, while Y is independent of it. As C/A varies, therefore, the corresponding point B in XYZ space will move along a straight line in a plane $Y = \text{const}$.

To describe this line, we begin with the limiting case $C/A = 0$.

When $C/A = 0$, Z vanishes, while X and Y take equal values; hence the initial point B' of the line lies in the first and third quadrants of the XY plane. The position of this point B' determines the position of the whole line relative to the SD. Recall (see Section 5) that these same quadrants contain the intersection of the SDSF and the XY plane. In addition, C/A may vary only from zero to 2, and so the motion of the point B along the straight line is restricted to a certain segment of finite length. The length and direction of this segment also influence the stability properties of the motion. The inclination of all such segments is determined by the derivative

$$dZ/dX = -1 - \frac{\delta}{\varepsilon(1 + \eta - k \operatorname{tg} \alpha)}$$

Obviously, as C/A increases the value of Z increases, X decreases, and the angle between the plane $Z = 0$ and the segments is greater than $\pi/4$ but at most $\pi/2$. Taking the properties of the family of segments into account, let us compare them with the family of level curves $Y = \text{const}$, which characterize the contour of the surface E (see Fig. 2). This will enable us to evaluate the effect of changes in C/A on the stability.

First let us consider the case in which the initial point B_1' of the segment lies in the first quadrant of the XY plane (Fig. 2).

Suppose first that B_1' is a point of the SD. It can be seen from an analysis of the relief of the surface E when $X > 0$, $Y > 0$, of its level curves and of the inclination of the segments, that as C/A increases the representative point B, moving away from the plane $Z = 0$, will approach E. Whether it will leave the SD or remain inside it along its entire path will depend on the totality of all the other parameters.

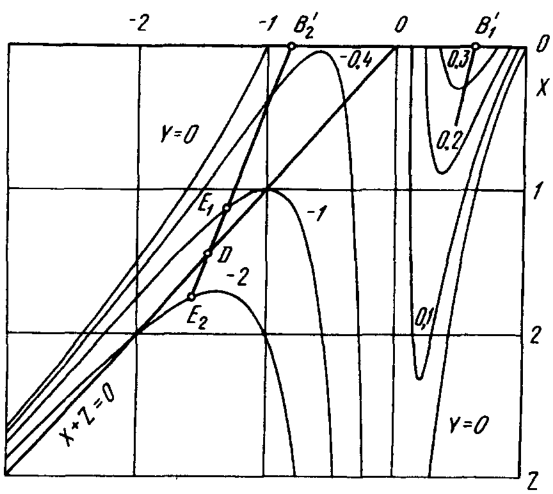

Fig. 2.

We will now discuss the case in which the initial points of the segments, as before, lie in the first quadrant of the XY plane, but outside the SD, i.e. above the boundary curve $Y = X - X^3$. In that situation the characteristic point will be a local maximum point of the aforementioned curve, with coordinates $X = 1/\sqrt{3}$, $Y = 2(3\sqrt{3})$. If, say, the initial point B_1' lies to the left of or above the maximum, i.e. the coordinates of B_1' satisfy the inequalities $Y > 2/(3\sqrt{3})$ or $X < 1/\sqrt{3}$, then for any values of C/A and the other parameters the segment will lie outside the SD. Otherwise, if $X > 1/\sqrt{3}$, $Y < 2(3\sqrt{3})$, an increase in C/A may result in a variety of situations. One possible case, as before, is a loss of stability. But there are other possibilities: stability may occur; stability may be maintained up to the maximum C/A or subsequently be lost for certain $C/A < 2$. A more definite picture requires numerical computations, taking all the other parameters into account.

Thus, if the initial point of the segment lies in the first quadrant of the XY plane, a change in the ratio C/A may affect stability in all possible ways.

The situation is different when the initial points of the segments lie in the third quadrant of the XY plane. We know [1] that increasing C/A from zero will always stabilize permanent rotation. In other words, the corresponding segment of the straight line, beginning at a point B_2' in the plane $Z = 0$, will always cut the plane $X + Z = 0$ at a point D (Fig. 2).

Here there are three possibilities.

If the initial point B_2' is in the SDSF, it will always remain there, but part of the segment will lie in the intersection of the two stability domains being compared.

The Y coordinate of B_2' may be chosen outside the stability domain in such a way that increasing C/A from zero to some value will stabilize steady fall, i.e. the point B will cut the surface E at a point E_1. Only after crossing the plane $X + Z = 0$, as a result of a further increase in C/A, will the point again enter the intersection of the stability domains.

Thus, in the two previous cases an increase in C/A guarantees stability, while for stability of permanent rotation one needs a value of C/A so large that the steady fall is certainly stable.

The opposite situation may also occur: for a certain choice of the Y coordinate of B_2' outside the SD, increasing C/A may guarantee stability of permanent rotation, i.e. the motion will reach the plane $X + Z = 0$ at the point D. Whether one can guarantee stability of steady fall by a further increase in C/A, i.e. reaching the point E_2, depends on the relief of the surface E in the direction of the segment and on the values of all the other parameters.

8. EVOLUTION OF STABILITY AS THE VANE ANGLE β VARIES

The stability of steady fall (2.1)–(2.4) depends not only on mass and geometrical data, as represented by the parameters ε, N and C/A, but also on the angle β at which the vane is mounted. By (2.3), this angle determines the steady-state angle of attack α_0 and, via the functions c_x and c_y, affects the quantities X, Y, Z.

We will use a device similar to that used in [1], considering the functions $X(\beta)$, $Y(\beta)$, $Z(\beta)$ as the parametric representation of a certain curve Σ in XYZ space. As β varies monotonically (and accordingly also α_0), the representative point B moves along Σ. Its position relative to the SD tells us something of the possible behaviour of the body.

When $\beta = 0$ and $\alpha_0 = 0$, we have $X = Y = Z = 0$, and hence the initial point of the curve Σ is the origin. As β and, by Eq. (2.3) α_0, increase, X, Y, Z may become positive. For example, at small α_0 values, the aerodynamic data of a flat circular disk [7] give $\xi > 2k$, guaranteeing that $X > 0$, $Y > 0$, $Z > 0$.

When $X > 0$, $Y > 0$, permanent rotation is unstable, but part of the SDSF nevertheless lies in that part of XYZ space; one is therefore interested in whether the curve Σ can pass through that zone. In other words, can the stability conditions for steady fall be satisfied at small α_0 values, when $\xi > 2k$? On the assumption that $b_2 > 0$, $b_3 > 0$, $X > 0$, $Y > 0$, let $\alpha_0 \to 0$ in equality (4.4). This yields the condition $a_2 b_2 > a_1 b_3$, which is equivalent to the requirement that the inclination of the curve Σ at the origin must be less than that of the surface E at the same point, and that at least the initial part of Σ must pass through the SD when $X > 0$, $Y > 0$. Substituting expressions (4.3) for a_i and b_i into this condition, it can be shown that it will always hold if $\varepsilon < [k'(0) - 2]/[k'(0) + 2]$. Hence it is clear that a suitable choice of ε will ensure the truth of this inequality.

For flat plates of many shapers [7, 8], the derivative $k'(\alpha)$ may considerably exceed unity at $\alpha_0 = 0$. In that case ε should not exceed a certain quantity close to unity. This requirement can always be fully satisfied.

Thus, over a range of angles of attack α_0 approaching $\alpha_0 = 0$, steady fall may be asymptotically stable, while the corresponding permanent rotation is always unstable.

We shall not limit ourselves to local properties of the curve Σ. As β and, accordingly, α_0 increase, the difference $\xi - 2k$ will begin to decrease. This will reduce X and Y. As a result, at some $\alpha_0 = \alpha_*$ the point B will leave the SD at $X > 0$, $Y > 0$, and then reach the plane $X = 0$ at some positive Y. When β and α_0 are increased further, the difference $\xi - 2k$ will vanish, so that Y will vanish. The curve Σ will intersect the plane $Y = 0$. By the conditions we have adopted, the sum $X + Z$ will always be positive at the point of intersection. Thus Σ will intersect the plane $Y = 0$ between the straight lines $X = 0$ and $X + Z = 0$ in the plane $Y = 0$. Since that is the location of the common part of the boundaries of the two SDs with which we are concerned, it follows that the curve Σ will subsequently lie at the intersection of the stability domains with $X < 0$, $Y < 0$, so that both steady fall and permanent rotation will become asymptotically stable at the same values of β and α_0.

As to further changes in the properties of the motion, the following remark is in order. It is well known [1] that when $C/A < 1$ and α_0 is increased, permanent rotation will become unstable at some $\alpha_0 = \alpha_{**}$. In XYZ space, the curve Σ will intersect the boundary of the SDPR—the plane $X + Z = 0$. The position of the point of intersection relative to the surface E will indicate whether the motion is stable. Let us subtract the ordinate y of the point at which Σ intersects the plane $X + Z = 0$ from the corresponding ordinate of the point on the straight line $X + Z = 0$, $X = Y$ along which E cuts the plane $X + Z = 0$. The sign of the difference will obviously determine whether the point of intersection of Σ and the plane $X + Z = 0$ belongs to the SDSF or not. It can be shown that the sign of the difference is precisely the sign of the quantity

$$2(k' - k^2 - k\,\mathrm{tg}\,\alpha_0) - \xi k(1 + 2k^2 + k\,\mathrm{tg}\,\alpha_0)$$

This quantity will certainly be negative if $k'(\alpha_{**}) \leqslant 0$. In that case the point in question on Σ will lie below E, i.e. outside the SDSF. In other words, as β and α_0 increase the point B will first cross E and only then reach the plane $X + Z = 0$.

Moreover, as can be verified by computations, for vanes in the shape of flat plates, the above difference is strictly negative over the entire range of angles of attack α_0 from $\alpha_0 = \alpha_*$ to $\alpha_0 = \alpha_{**}$, i.e. from the intersection of Σ with the plane $Y = 0$ to its intersection with the plane $X + Z = 0$.

Thus, permanent rotation will become unstable before the onset of instability in the corresponding steady fall.

At the same time, for sufficiently small C/A values, even with vanes with a relatively poor lift–drag ratio, the initial part of the curve Σ may lie in the SDSF for $X < 0$ and $Y < 0$, but outside the SDPR.

9. TYPES OF BEHAVIOUR OF THE BODY WHEN THERE IS LOSS OF STABILITY IN STEADY FALL

The loss of stability across the boundary of the SD may vary in nature, depending on where the crossing occurs. Suppose that the parameters of the system are such that the representative point B in XYZ space is located near the part of the boundary in the plane $Y = 0$ outside the

SD, i.e. where $XY < 0$. Ignoring quantities of second and higher order of smallness, we determine the root μ of the characteristic equation (4.2) of smallest absolute value

$$\mu = -b_3(b_2 + ia_2) / \sqrt{a_2^2 + b_2^2}$$

If the point B lies near the common part of the boundaries of the SDPR and SDSF, i.e. $X < 0$, both types of motion will become unstable owing to the development of slow reverse precession. The loss of stability will thus occur in the same way in both types of motion.

If the point B is outside the SDSF, with $X > 0$, and $Y < 0$, slow precession will develop in the same sense as the rotation.

Another type of motion will take place if B lies on the part of the surface E bounding the SDSF. The characteristic equation (4.2) will then have one purely imaginary root $\mu = -ib_2 / a_1$. Since $a_1 > 0$, $b_2 b_3 > 0$, the other two roots will have negative real parts. Loss of stability will be due to the development of fast precession, as in the destabilization of permanent rotation, but the angular velocity of the precession will not be equal to that of Euler precession.

If one considers the part of E with $X > 0$ and $Y > 0$, the sign of the angular velocity of precession of the body in free fall is the reverse of that of Euler precession.

In the neighbourhood of the other part of E, where $X < 0$, $Y < 0$, the sign of the angular velocity of fast precession will be the same as that of Euler precession. But if, additionally, the parameter ε tends to zero, then the two velocities will also be similar in absolute value—yet another illustration of the role of ε as a measure of similarity between the two types of motion.

Thus, by comparing the stability conditions for these two, so to speak, "aerodynamically similar" types of motion of a body, we have been able to establish several useful properties. First, we have determined some common features of the behaviour of the body, including the fact that the boundaries of the stability domains coincide in part. Second, we have pointed out differences in their behaviour and proposed parameters that measure these differences. This information may prove useful in interpreting the results of wind-tunnel experimentation with mock-ups of aircraft and in using them to predict the latter's free motion.

An important role was played in the analysis by a universal geometrical representation—the SD of a linear mechanical system with three degrees of freedom, possessing a well-defined symmetry—in the parameter space of its characteristic equation.

REFERENCES

1. PRIVALOV V. A. and SAMSONOV V. A., The stability of motion of a body autorotating in the flow of a medium. *Izv. Akad. Nauk SSSR MTT* 2, 32–38, 1990.
2. CHEBOTAREV N. G. and MEIMAN N. N., The Routh–Hurwitz problem for polynomials and entire functions. *Trudy Mat. Inst. Steklova* **26**, 1–332, 1949.
3. FEDYAYEVSKII K. K. and BLYUMINA L. Kh., *Hydrodynamics of Detached Flow Around Bodies*. Mashinostroyeniye, Moscow, 1977.
4. LOKSHIN B. Ya., PRIVALOV V. A. and SAMSONOV V. A., *Introduction to the Problem of the Motion of a Body in a Resistant Medium*. Izd. Mosk. Gos. Univ., 1986.
5. ZENKIN A. N., PRIVALOV V. A. and SAMSONOV V. A., A quasistatic model of the action of a medium on an autorotating body. *Izv. Ross. Akad. Nauk, MTT* 4, 73–78, 1993.
6. HESS F., The aerodynamics of boomerangs. *Scientific American* **219**, 5, 124–136, November, 1968.
7. FLACHSBART O., Messungen an ebenen und gewölbten Platten. In *Ergebnisse der aerodynamischen Versuchsanstalt zu Göttingen*, Vo. 4, pp. 96–100. Prandtl & Betz, Munich, 1932.
8. TABACHNIKOV V. G., Stationary characteristics of wings at low velocities in the entire range of angles of attack. *Trudy TsAGI* 1621, 79–93, 1974.
9. NOVOZHILOV I. V., *Fractional Analysis*. Izd. Mosk. Gos. Univ., Moscow, 1991.

Translated by D.L.